J. Differential Equations 253 (2012) 933–950

Contents lists available at SciVerse ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

Global solvability for first order real linear partial differential operators II ☆

José R. dos Santos Filho [a], Maurício Fronza da Silva [b,*]

[a] Departamento de Matemática, Universidade Federal de São Carlos, São Carlos, SP, Brazil
[b] Departamento de Matemática, Universidade Federal de Santa Maria, Santa Maria, RS, Brazil

ARTICLE INFO

Article history:
Received 9 June 2011
Revised 17 February 2012
Available online 7 May 2012

MSC:
35A05
35F05

Keywords:
First order partial differential operators
Global solvability
Global existence of first order partial differential operators

ABSTRACT

Let L be a real C^∞ vector field on a smooth manifold X, vanishing at exactly one point x_0. From the pioneering work of B. Malgrange (1955–1956) [6], we know that solvability of $P = L + c$ on $C^\infty(X)$, for $c \in C^\infty(X, \mathbb{C})$, implies that: (a) X is L-convex. Also, it follows: (b) a non-resonance condition for the jet-solvability at x_0.

In a previous paper, in addition to (a) and (b), the authors showed that P is globally solvable on C^∞ if we assume: (c) a non-resonance condition in order to linearize L near x_0; that (d) the only relatively compact orbit of L is $\{x_0\}$; and that (e) c is real.

Here we obtain the same conclusion without (c) and (e).

1. Introduction

Let X be a C^∞ manifold with a countable basis of open sets and $P : C^\infty(X) \to C^\infty(X)$ a linear partial differential operator. P is said **globally solvable**, or **solvable**, on $C^\infty(X)$ when $P(C^\infty(X)) = C^\infty(X)$.

Unless otherwise mentioned, from now on we assume that P has real principal symbol and is of order one. We write $P = L + c$ where L is a real vector field and c is a smooth complex function.

Suppose that $X = \mathbb{R}^n$ and that the coefficients of L are linear functions. If the origin is a hyperbolic critical point of L and $c \in \mathbb{C}$, V. Guillemin and D. Schaeffer [3] showed that the equation $Pu = f$ has a C^∞ solution in a neighborhood of zero, for $f \in C^\infty(\mathbb{R}^n)$ flat at the origin. In the case when the coefficients are not linear, using S. Sternberg's linearization theorem, see [11] and [8], the same conclusion is true for an isolated critical point x_0. The condition of linearization is

☆ With appendix by Waldeck Schützer.

* Corresponding author.
E-mail addresses: santos@dm.ufscar.br (J.R. dos Santos Filho), mfronza@smail.ufsm.br (M.F. da Silva).

0022-0396/$ – see front matter
http://dx.doi.org/10.1016/j.jde.2012.04.019

$$\lambda_j \neq \sum_{k=1}^{n} m_k \lambda_k, \quad j \in \{1, 2, \ldots, n\},\ m_1, \ldots, m_n \in \mathbb{N},\ \sum_{k=1}^{n} m_k \geqslant 2, \tag{NRC 1}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $DL(x_0)$.

A hypothesis like (NRC 1) is usually called a non-resonance condition. Under this hypothesis we have that the origin is hyperbolic but the converse is not true.

To solve the equation $Pu = f$ near x_0 for an arbitrary $f \in C^\infty(X)$, in [9] we need to consider the following additional non-resonance condition

$$-c(x_0) \neq \sum_{j=1}^{n} m_j \operatorname{Re} \lambda_j, \quad m_1, \ldots, m_{n'} \in \mathbb{N},\ m_{n'+1}, \ldots, m_n \in 2\mathbb{N}. \tag{NRC 2}$$

Here $\lambda_1, \lambda_2, \ldots, \lambda_{n'}$ are the real eigenvalues of $DL(x_0)$ and $\lambda_{n'+1}, \ldots, \lambda_n$ are the non-real eigenvalues.

We say that X **is convex with respect to the trajectories of** L if $\forall K \Subset X$, $\exists K' \Subset X$ such that any compact interval of the trajectory of L with endpoints in K, is contained in K'. See [2] and [6] for motivation. As usual $F \Subset Y$ means that F is a compact subset of Y.

In [9] we prove the following result:

Theorem 1.1. *Let $P = L + c$ be a first order differential operator with coefficients in $C^\infty(X, \mathbb{R})$ such that L has a critical point at x_0. If*

(a) (NRC 1) *and* (NRC 2) *are valid,*
(b) *no orbit of L on $X \setminus \{x_0\}$ is relatively compact in X, and*
(c) *X is convex with respect to the trajectories of L*

then

$$P \text{ is globally solvable on } C^\infty(X).$$

When L has no critical point, conditions (b) and (c) are equivalent to conditions (d.1) and (d.2) of Theorem 6.4.2 in [2].

In the present work, starting from the local parametrization of the invariant submanifolds used by P. Hartman [4], we extend the Guillemin–Schaeffer's result for an arbitrary vector field with a hyperbolic critical point and $c \in C^\infty(X)$. Hence we can solve the equation $Pu = f$ near x_0, for all $f \in C^\infty(\mathbb{R}^n)$ which are flat at x_0, even when (NRC 1) is false.

Also we replace (NRC 2) with a new non-resonance condition in such way to consider the complex valued lower order term. In fact, that condition is obtained directly from the formal Taylor's expansion. This is summarized in

Lemma 1.1. *Suppose that L is a real vector field on $\mathbb{R}^n$ with a critical point x_0, $c \in C^\infty(\mathbb{R}^n)$ and $P = L + c$. The condition*

$$-c(x_0) \notin \sum_{k=1}^{n} m_k \operatorname{Re} \lambda_k \pm i \sum_{k=1}^{(n-n')/2} \mathscr{M}(m_{n'+2k-1} + m_{n'+2k}) \operatorname{Im} \lambda_k \tag{NRC}$$

for $m_1, \ldots, m_n \in \mathbb{N}$, where

$$\mathscr{M}(m) = \begin{cases} \{0, 2, \ldots, m\}, & \text{if } m \text{ even}, \\ \{1, 3, \ldots, m\}, & \text{if } m \text{ odd}, \end{cases}$$

is equivalent to: for all $f \in C^\infty(\mathbb{R}^n)$ there exists $u \in C^\infty(\mathbb{R}^n)$ such that $Pu - f$ is flat at x_0.

Our main theorem, which is a generalization of Theorem 1.1, is:

Theorem 1.2. *Let $P = L + c$ be a first order differential operator with coefficients in $C^\infty(X)$ such that L is a real vector field with a hyperbolic critical point at x_0 and c is a (real or complex) function. If*

(a) (NRC) *is valid,*
(b) *no orbit of L on $X \setminus \{x_0\}$ is relatively compact in X, and*
(c) *X is convex with respect to the trajectories of L*

then

$$P \text{ is globally solvable on } C^\infty(X).$$

Notice that when c is a real function, the condition (NRC) coincides with (NRC 2). The hypotheses of Theorem 1.2 have consequences on global stability for local attractors. More precisely,

Remark 1.1. Suppose that X is a connected manifold and that x_0 is a local hyperbolic attractor of L. Then the following conditions are equivalent:

(i) (b) and (c) of Theorem 1.2 hold.
(ii) $\{x_0\}$ is a global attractor of L.

Suppose (i) is valid. We will see that the boundary $\partial\mathcal{B}(x_0)$ of the basin of attraction $\mathcal{B}(x_0) = \{x \in X;\ \lim_{t\to\omega_+(x)} \gamma(t,x) = x_0\}$ of $\{x_0\}$ is empty. Here γ is the flow of L and we denote the maximal interval of it at $x \in X$ by $(\omega_-(x), \omega_+(x))$.

Suppose that $x \in \partial\mathcal{B}(x_0)$. Since x_0 is a local attractor of L, $\mathcal{B}(x_0)$ is an open subset of X, hence, $x \notin \mathcal{B}(x_0)$. Consider neighborhoods U_x of x and U_{x_0} of x_0 such that $\overline{U_x}, \overline{U_{x_0}} \Subset X$. Here $\overline{F}$ denotes the closure of the subset F. Take $K = \overline{U_{x_0}} \cup \overline{U_x}$. It is easy to see that for such K there is no compact K' as in the definition of convexity of X with respect to the trajectories of L, which contradicts (c). This follows at once from the fact that (b) implies that the ω-limit of the trajectory $\gamma(\cdot, x)$ cannot be relatively compact in X.

Conversely, the implication (ii) $\Rightarrow$ (b) is valid. In fact, from (ii) we have that any trajectory has ω-limit equal to $\{x_0\}$. Also its α-limit is empty, for otherwise a non-trivial loop would be a trajectory with endpoints at $\{x_0\}$, contradicting the hypothesis that x_0 is a local hyperbolic attractor of L.

To prove that (ii) $\Rightarrow$ (c), first we choose a neighborhood U of x_0, such that its closure $\overline{U} \Subset X$ is positively invariant by the flow and a sequence $\{K_j\}$ of compact subsets of X with the following properties: $\bigcup K_j = X$ and $K_j \subset K_{j+1}^\circ$, $j = 1, 2, \ldots,$ here we denote A° as the interior of $A \subset X$. If (c) is false then there exists $K \Subset X$, a sequence $\{[x_j, x_j']\}$ of compact trajectories segments with endpoints in K and a sequence $\{y_j\}$ such that $y_j \in [x_j, x_j']$ but $y_j \notin (K_j \cup U)$, $j = 1, 2, \ldots.$ By (ii), without loss of generality, we can assume that $x_j' \in \overline{U}$. From compactness, we have a converging subsequence of $\{x_j\}$ to a point $x \in K$. Then we can see that once the trajectory starting at x enters U it never leaves that neighborhood. This contradicts $y_j \notin K_j$ for large j, due to the continuous dependence.

As in the previous work, first we solve the equation $Pu = f$ near the critical point then we use Theorem 6.4.2 of [2] to solve the equation outside a neighborhood of the critical point. The first step is to generalize Theorem 2 of [3]:

Theorem 1.3. *Let $P = L + c$ be a first order differential operator with C^∞ coefficients on $\mathbb{R}^n$, where L is a real vector field such that the origin is a hyperbolic critical point. If $f \in C^\infty(\mathbb{R}^n)$ vanishes of infinite order at the origin then there exists $u \in C^\infty(\mathbb{R}^n)$ vanishing of infinite order at the origin such that $Pu = f$ in a neighborhood of the origin.*

As we will show, the proof of Theorem 1.3 follows easily from the next result, which plays the same role as Theorem 4 of [3] does in the proof of Theorem 2 of [3].

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n$ and $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$. Hence $\partial_j f = \frac{\partial}{\partial x_j} f$, $j = 1, 2, \ldots, n$ and $\partial^\alpha f = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} f$. The Euclidean distance of $x \in \mathbb{R}^n$ to a subset F of $\mathbb{R}^n$ is denoted by $|x - F|$.

Theorem 1.4. *Let $L = \sum_{k=1}^n a_k \partial_k$ be a smooth real vector field on $\mathbb{R}^n$ such that for each $m = 1, 2, \ldots$ there exists $M_m > 0$ such that*

$$\left|\partial^\alpha a_k(x)\right| \leqslant M_m, \quad x \in \mathbb{R}^n,\ k = 1, 2, \ldots, n,\ \alpha \in \mathbb{N}^n,\ 0 < |\alpha| \leqslant m.$$

Suppose that $c, f \in C_0^\infty(\mathbb{R}^n)$ and N is a closed subset of $\mathbb{R}^n$ invariant under the flow γ of L and f vanishing of infinite order at N. Suppose that $M, \mu > 0$ and $E \subset \mathbb{R}^n$ is a linear subspace invariant under γ such that

$$\left|\gamma(t, x) - N\right| \leqslant M e^{-\mu t} |x - N|, \quad t \geqslant 0,\ x \in E. \tag{1.1}$$

Then there exists $u \in C^\infty(\mathbb{R}^n)$ such that $Pu - f$ vanishes of infinite order at E. Furthermore, u vanishes of infinite order at N.

For recent applications of globally solvable first order linear partial differential operators we refer to [1] and [10].

This paper is organized in the following way: in the second and third sections, we will present estimates for the flow near the unstable manifold and from the derivatives of the flow. Finally, in the fourth section, using the previous sections, we will prove the theorems. The proof of Lemma 1.1 is presented in Appendix A, it follows quite closely the proof of Lemma 5 of [9]. In Appendix B, using a more appropriate technique, an alternative proof of this fact is given by our colleague Waldeck Schützer (DM-UFSCar).

2. Estimates for the flow near the unstable manifold

Let $M(n, \mathbb{R})$ be the space of real matrices $n \times n$. $A \in M(n, \mathbb{R})$ is said hyperbolic if the real part of its eigenvalues are non-zero. Here we recall the following elementary result:

Lemma 2.1. *Assume that $A \in M(n, \mathbb{R})$ is hyperbolic. Let γ be the flow of the linear vector field $x \mapsto Ax$, $x \in \mathbb{R}^n$. If the eigenvalues of A have negative real part then there are positive real numbers M_A and ρ_A such that*

$$\left|\gamma(t, x)\right| \leqslant M_A e^{-\rho_A t} |x|, \quad t \geqslant 0,\ x \in \mathbb{R}^n.$$

Remark 2.1. From Lemma 2.1 it follows that, if the eigenvalues of A have positive real part then there are positive real numbers M_A and ρ_A such that

$$\left|\gamma(t, x)\right| \leqslant M_A e^{\rho_A t} |x|, \quad t \leqslant 0,\ x \in \mathbb{R}^n.$$

From Gronwall's Lemma it follows a local extension of Lemma 2.1 for C^1 vector fields of $\mathbb{R}^n$ with a local attractor at the origin.

In this section L will be a real vector field with $C^\infty(\mathbb{R}^n)$ coefficients and with a hyperbolic critical point at x_0. We denote the stable (resp. unstable) manifold of L at x_0 by $W^s(x_0)$ (resp. $W^u(x_0)$). Let s be the number of the eigenvalues of $DL(x_0)$ with negative real part. Here $x = (x_1, x_2)$, with $x_1 \in \mathbb{R}^s$ and $x_2 \in \mathbb{R}^{n-s}$.

The next result implies that $W^s(x_0)$ (resp. $W^u(x_0)$) is an immersed submanifold of $\mathbb{R}^n$ [7].

Lemma 2.2. *Suppose that x_0 is a hyperbolic critical point of L. Then there exist $r > 0$ and $g \in C^\infty(x_0 + B^s(r), B^{n-s}(r))$ such that $g(x_0) = 0$, $Dg(x_0) = 0$ and the stable manifold of L near x_0 is $x_0 + G(g)$.*

Here $B^m(r)$ is the open ball of $\mathbb{R}^m$ centered at the origin with radius r, and $G(g)$ is the graph of g identified with a subset of $\mathbb{R}^n$.

In order to obtain estimates for the flow of L near the unstable manifold we define a special class of vector fields. Let $\mathcal{V}$ be the set of vector fields of the form

$$\widetilde{L}(x) = Ax + \big(F_1(x), F_2(x)\big), \quad x \in \mathbb{R}^n,$$

where A and $F = (F_1, F_2)$ have the following properties:

(P.1) $A \in M(n, \mathbb{R})$ and $A = \left[\begin{smallmatrix} P & 0 \\ 0 & Q \end{smallmatrix}\right]$, where $P \in M(s, \mathbb{R})$, $Q \in M(n-s, \mathbb{R})$, the real part of eigenvalues of P are negative and the real part of the eigenvalues of Q are positive,

(P.2) (a) $F_1 \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^s)$, $F_2 \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n-s})$, $F_i(0) = 0$, $DF_i(0) = 0$, $i = 1, 2$,
(b) $|DF_i| \leqslant \min\{\frac{\rho_P}{2M_P}, \frac{\rho_Q}{2M_Q}\}$, $i = 1, 2$, and
(c) $F_i(x_1, x_2) = 0$ if $x_i = 0$, $i = 1, 2$.

Here ρ_P, M_P, ρ_Q and M_Q are given by Lemma 2.1 and Remark 2.1, for P and Q, respectively.

Remark 2.2. Every $\widetilde{L} \in \mathcal{V}$ is a vector field on the Lipschitz class, so the flow of $\widetilde{L}$ is globally defined.

The next result is an extension of Lemma 2.1 for non-linear vector fields.

Lemma 2.3. *If $\widetilde{L} \in \mathcal{V}$ then*

$$W^s(0) = \mathbb{R}^s \times \{0''\}, \quad 0'' \in \mathbb{R}^{n-s}; \qquad W^u(0) = \{0'\} \times \mathbb{R}^{n-s}, \quad 0' \in \mathbb{R}^s;$$

and there are $M, \mu > 0$ such that

$$\big|\gamma(t,x) - W^u(0)\big| \leqslant Me^{-\mu t}\big|x - W^u(0)\big|, \quad t \geqslant 0,\ x \in \mathbb{R}^n. \tag{2.1}$$

Proof. Let $\gamma = (\gamma_1, \gamma_2)$ be the flow of $\widetilde{L}$ and $x = (x_1, x_2)$ be an arbitrary point of $\mathbb{R}^n$. We have

$$\gamma_1(t,x) = e^{Pt}x_1 + \int_0^t e^{P(t-\tau)} F_1\big(\gamma(\tau,x)\big)\, d\tau, \quad t \in \mathbb{R}$$

and

$$\gamma_2(t,x) = e^{Qt}x_2 + \int_0^t e^{Q(t-\tau)} F_2\big(\gamma(\tau,x)\big)\, d\tau, \quad t \in \mathbb{R},$$

then

$$\big|\gamma_1(t,x)\big| \leqslant \big|e^{Pt}x_1\big| + \int_0^t \big|e^{P(t-\tau)} F_1\big(\gamma(\tau,x)\big)\big|\, d\tau, \quad t \in \mathbb{R}.$$

Define $\rho = \min\{\rho_P, \rho_Q\}$ and $M = \max\{M_P, M_Q\}$. From (P.1) and Lemma 2.1 we obtain

$$\big|\gamma_1(t,x)\big| \leqslant Me^{-\rho t}|x_1| + \int_0^t Me^{-\rho(t-\tau)}\big|F_1\big(\gamma(\tau,x)\big)\big|\, d\tau, \quad t \geqslant 0.$$

From (P.2) and the mean value inequality it follows that

$$\big|F_1\big(\gamma(\tau,x)\big)\big| = \big|F_1\big(\gamma_1(\tau,x),\gamma_2(\tau,x)\big) - F_1\big(0',\gamma_2(\tau,x)\big)\big| \leqslant \frac{\rho}{2M}\big|\gamma_1(\tau,x)\big|,$$

and hence

$$\big|\gamma_1(t,x)\big| \leqslant Me^{-\rho t}|x_1| + \int_0^t \frac{\rho}{2}e^{-\rho(t-\tau)}\big|\gamma_1(\tau,x)\big|\,d\tau, \quad t \geqslant 0.$$

Therefore

$$e^{\rho t}\big|\gamma_1(t,x)\big| \leqslant M|x_1| + \int_0^t \frac{\rho}{2}e^{\rho\tau}\big|\gamma_1(\tau,x)\big|\,d\tau, \quad t \geqslant 0.$$

From Gronwall's Lemma it follows that

$$\big|\gamma_1(t,x)\big| \leqslant Me^{-\frac{\rho}{2}t}|x_1|, \quad t \geqslant 0,$$

and similarly

$$\big|\gamma_2(t,x)\big| \leqslant Me^{\frac{\rho}{2}t}|x_1|, \quad t \leqslant 0.$$

These inequalities imply that $\mathbb{R}^s \times \{0''\} \subset W^s(0)$ and $\{0'\} \times \mathbb{R}^{n-s} \subset W^u(0)$. Since that $W^s(0)$ and $W^u(0)$ are immersed submanifold with dimension s and $n-s$, respectively, the converse inclusions also hold. □

Remark 2.3. The proof of Lemma 2.3 shows that if $\widetilde{L} \in \mathcal{V}$ then there exist $M, \mu > 0$ such that

$$\big|\gamma(t,x)\big| \leqslant Me^{-\mu t}|x|, \quad t \geqslant 0,\ x \in W^s(0).$$

Proposition 2.1. *Let L be a real C^∞ vector field in $\mathbb{R}^n$. Suppose that x_0 is a hyperbolic critical point for L. Then there are a neighborhood $U \subset \mathbb{R}^n$ of the origin, a diffeomorphism $h \in C^\infty(x_0+U, U)$ and a vector field $\widetilde{L} \in \mathcal{V}$ such that $Dh_{h^{-1}(y)}L(h^{-1}(y)) = \widetilde{L}(y)$, $\forall y \in U$.*

Proof. Without lost of generality we assume that $x_0 = 0$. Consider $A = DL(0)$ and take $F = (F_1, F_2) \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ such that

$$L(x) = Ax + F(x), \quad x \in \mathbb{R}^n, \tag{2.2}$$

therefore (P.2) (a) holds.

We will divide the proof in three steps.

Step 1. There exists a linear change of variables such that the matrix A of (2.2) becomes a matrix with the property (P.1), so that the new F still has the property (P.2) (a).

In fact, consider the change of variables which transform A into its the real Jordan form.

Step 2. If L satisfies (P.1) and (P.2) (a) then there are a neighborhood $U \subset \mathbb{R}^n$ of the origin and a C^∞ diffeomorphism $h: U \to U$ such that the vector field $Dh_{h^{-1}(y)}L(h^{-1}(y))$ satisfies (P.1), (P.2) (a) and (P.2) (c) in U.

In fact, from Lemma 2.2 there exist $r > 0$ and $g \in C^\infty(B^{n-s}(r), B^s(r))$ such that $g(0'') = 0'$, $Dg(0'') = 0'$ and the unstable manifold of L near the origin is the graph of g. Define $U = B^s(r) \times B^{n-s}(r)$.

Consider the function $h : U \to U$ given by

$$\begin{aligned} y_1 &= x_1 - g(x_2), \\ y_2 &= x_2. \end{aligned} \tag{2.3}$$

Notice that h is a diffeomorphism, since h^{-1} is given by

$$\begin{aligned} x_1 &= y_1 + g(y_2), \\ x_2 &= y_2. \end{aligned}$$

Moreover we have

$$Dh(x) = \begin{pmatrix} I_s & -Dg(x_2) \\ 0 & I_{n-s} \end{pmatrix},$$

hence

$$\widetilde{L}(y) := Dh_{h^{-1}(y)} L\big(h^{-1}(y)\big) = Ay + \big(G_1(y), G_2(y)\big), \quad y \in U,$$

where $G_1 \in C^\infty(U, \mathbb{R}^s)$ and $G_2 \in C^\infty(U, \mathbb{R}^{n-s})$ are given by

$$G_1(y) = Pg(y_2) + F_1 \circ h^{-1}(y) - Dg(y_2)\big(Q y_2 + F_2 \circ h^{-1}(y)\big)$$

and

$$G_2(y) = F_2 \circ h^{-1}(y).$$

Since $g(0'') = 0'$, $Dg(0'') = 0'$, $h(0) = 0$ and F_1, F_2 have the property (P.2) (a), it follows that G_1, G_2 have the property (P.2) (a). Since $W^u(0)$ is the graph of g, from (2.3) it follows that $h^{-1}(0', y_2) \in W^u(0)$. Hence $(\{0'\} \times \mathbb{R}^{n-s}) \cap U$ is contained in the unstable manifold of $\widetilde{L}$ at the origin, which has dimension $n-s$. Then the unstable manifold of $\widetilde{L}$ at the origin is $(\{0'\} \times \mathbb{R}^{n-s}) \cap U$. Since this manifold is invariant under the flow of $\widetilde{L}$, it follows that $G_1(0', y_2) = 0$.

In the same way, under a change of variables in x_2 we conclude Step 2.

Step 3. Let L be a real vector field in a neighborhood $U \subset \mathbb{R}^n$ of the origin. If L has the properties (P.1), (P.2) (a) and (P.2) (c) in U then L has an extension to $\mathbb{R}^n$ which belongs to $\mathcal{V}$.

In fact, this can be done by using cutoff functions (see [7]). □

3. Estimates for the derivatives of the flow

The basic estimate in this section is given by the following result [4]:

Lemma 3.1. *Suppose that $A : I \to M(n, \mathbb{R})$ and $b : I \to \mathbb{R}^n$ are continuous functions defined on the compact interval $I \subset \mathbb{R}$. If $t_0 \in I$ and $y : I \to \mathbb{R}^n$ is a solution of*

$$\begin{cases} y' = A(t)y + b(t), \\ y(t_0) = y_0 \end{cases}$$

in I then

$$\left|y(t)\right| \leqslant \left\{|y_0| + \int_{t_0}^{t} \left|b(s)\right| ds\right\} \exp \int_{t_0}^{t} \left|A(s)\right| ds, \quad t \in I.$$

This estimate holds for either the Euclidean or the maximum norm.

Suppose that $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ in $\mathbb{N}^n$. We say that $\alpha < \beta$ when $\alpha_j \leqslant \beta_j$, $\forall j \in \{1, 2, \ldots, n\}$ and for some j_0 we have that $\alpha_{j_0} < \beta_{j_0}$. For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ we denote $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. We consider $(\partial^\beta f)_{0<\beta<\alpha}$ the vector which its components are the derivatives of f with respect to x of order β, where $0 < \beta < \alpha$, in a fixed order.

The proof of the next result follows at once from the chain rule.

Lemma 3.2. *Let $L = \sum_{k=1}^{n} a_k \partial_k$ be a real smooth vector field and γ be the flow of L. For each $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{N}^n$, with $\alpha \neq 0$, we have that $\partial^\alpha \gamma$ satisfies a linear system of ordinary differential equations of the type*

$$\frac{d}{dt} \partial^\alpha \gamma(t, x) = DL\big(\gamma(t, x)\big) \partial^\alpha \gamma(t, x) + p_\alpha(t, x), \quad t \in \big(\omega_-(x), \omega_+(x)\big), \tag{3.1}$$

where p_α is a polynomial map of degree $\leqslant |\alpha|$ on the variables $(\partial^\beta \gamma)_{0<\beta<\alpha}$ whose coefficients depend on the derivatives of a_k's of order $\geqslant 1$ and $\leqslant |\alpha|$. Moreover, $p_\alpha \equiv 0$ when $|\alpha| = 1$.

Lemma 3.3. *Let $L = \sum_{k=1}^{n} a_k \partial_k$ be a smooth vector field. Suppose that for each $m = 1, 2, \ldots$ there exists $M_m > 0$ such that*

$$\left|\partial^\alpha a_k(x)\right| \leqslant M_m, \quad x \in \mathbb{R}^n,\ k = 1, 2, \ldots, n,\ 0 < |\alpha| \leqslant m. \tag{3.2}$$

Then for all $m \in \{1, 2, \ldots\}$ there exists $K_m > 0$ such that

$$\left|\partial^\alpha \gamma(t, x)\right| \leqslant e^{K_m t}, \quad t \geqslant 0,\ x \in \mathbb{R}^n,\ 0 < |\alpha| \leqslant m, \tag{3.3}$$

where $\gamma(t, x)$ is the flow of L and $|x| = \max\{|x_1|, |x_2|, \ldots, |x_n|\}$, $x \in \mathbb{R}^n$.

Proof. We will prove the result using induction on m. Suppose $m = 1$. Given $j = 1, 2, \ldots, n$, from Lemma 3.1 and Lemma 3.2 it follows that

$$\left|\partial_j \gamma(t, x)\right| \leqslant \left|\partial_j \gamma(0, x)\right| e^{M_1 t}, \quad t \geqslant 0,\ x \in \mathbb{R}^n.$$

Since $\gamma(0, x) = x$, $\forall x \in \mathbb{R}^n$, the conclusion follows if we choose $K_1 = M_1$.

Suppose that the result holds for all $\alpha \in \mathbb{N}^n$ such that $0 < |\alpha| \leqslant m$. Now, given $\alpha \in \mathbb{N}^n$ such that $|\alpha| = m + 1$, the idea of the proof is: first use Lemma 3.2 to obtain a suitable estimate to p_α. More precisely, prove that there exists $G_{m+1} > 0$ such that

$$\left|p_\alpha(t, x)\right| \leqslant G_{m+1} e^{(m+1) K_m t}, \quad t \geqslant 0,\ x \in \mathbb{R}^n. \tag{3.4}$$

Then complete the proof using Lemma 3.1.

In fact, from Lemma 3.2 it is enough to prove (3.4) for each monomial q on the variables $(\partial^\beta f)_{0<\beta<\alpha}$, such that each component of p_α satisfies an estimate as in (3.4). From the definition of p_α in Lemma 3.2 and (3.2) it follows that

$$\left|q(t, x)\right| \leqslant M_{m+1} \left|\left(\partial^{\beta^1} \gamma_{k_1}(t, x), \ldots, \partial^{\beta^j} \gamma_{k_j}(t, x)\right)^{\alpha^1}\right|, \quad t \geqslant 0,\ x \in \mathbb{R}^n,$$

where j is a suitable natural number, $k_1, k_2, \dots, k_j \in \{1, 2, \dots, n\}$, $\beta^1, \beta^2, \dots, \beta^j \in \mathbb{N}^n$, are such that

$$0 < \beta^1, \beta^2, \dots, \beta^j < \alpha \tag{3.5}$$

and $\alpha^1 \in \mathbb{N}^j$ is such that $|\alpha^1| \leqslant m+1$. From (3.5) and induction hypothesis it follows that

$$\left|q(t,x)\right| \leqslant M_{m+1} e^{|\alpha^1| K_m t}, \quad t \geqslant 0,\ x \in \mathbb{R}^n.$$

Since $|\alpha^1| \leqslant m+1$, this proves (3.4).

From Lemma 3.1 we now obtain

$$\left|\partial^\alpha \gamma(t,x)\right| \leqslant \left\{ \left|\partial^\alpha \gamma(0,x)\right| + \int_0^t \left|p_\alpha(s,x)\right| ds \right\} e^{M_1 t}, \quad t \geqslant 0,\ x \in \mathbb{R}^n.$$

Notice that the first term on the right is zero because $\gamma(0,x) = x$, $\forall x \in \mathbb{R}^n$ and $|\alpha| = m+1 \geqslant 2$. Then, the estimate (3.4) implies that

$$\left|\partial^\alpha \gamma(t,x)\right| \leqslant G_{m+1} e^{M_1 t} \int_0^t e^{(m+1)K_m s} ds, \quad t \geqslant 0,\ x \in \mathbb{R}^n.$$

The proof is finished if we choose $K_{m+1} = (m+1)K_m + M_1 + G_{m+1}$. □

4. Proofs of the theorems

In this section we will prove our main results.

Proof of Theorem 1.4. Consider

$$g(t,x) = \int_0^t c\big(\gamma(s,x)\big) ds, \quad t \geqslant 0,\ x \in \mathbb{R}^n \tag{4.1}$$

and

$$v(t,x) = -\int_0^t e^{g(s,x)} f\big(\gamma(s,x)\big) ds, \quad t \geqslant 0,\ x \in \mathbb{R}^n. \tag{4.2}$$

Observe that $g, v \in C^\infty(\mathbb{R}^+ \times \mathbb{R}^n)$. Since $c \in C_0^\infty(\mathbb{R}^n)$, it follows that there exists $\rho_0 > 0$ such that

$$\left|g(t,x)\right| \leqslant \rho_0 t, \quad t \geqslant 0,\ x \in \mathbb{R}^n. \tag{4.3}$$

We will divide the rest of the proof in four steps.

Step 1. For each $m = 1, 2, \dots$, there exists $K_m > 0$ such that

$$\left|\partial^\alpha g(t,x)\right| \leqslant e^{K_m t}, \quad t \geqslant 0,\ x \in \mathbb{R}^n,\ 0 < |\alpha| \leqslant m. \tag{4.4}$$

In fact, given $\alpha \in \mathbb{N}^n$ with $\alpha \neq 0$, by the chain's rule it follows that $\partial^\alpha(c \circ \gamma)$ is a polynomial map of degree $\leqslant |\alpha|$ on the variables $(\partial^\beta \gamma)_{0<\beta\leqslant\alpha}$ whose coefficients depend on the derivatives of c of order $\geqslant 1$ and $\leqslant |\alpha|$.

Since $c \in C^{\infty}(\mathbb{R}^n)$ and using Lemma 3.3, in the same way as in the proof of (3.4), we obtain that there exist $\rho_m, K'_m > 0$ such that

$$\left|\partial^{\alpha} c\big(\gamma(t,x)\big)\right| \leqslant \rho_m e^{K'_m t}, \quad t \geqslant 0,\ x \in \mathbb{R}^n.$$

Then

$$\left|\partial^{\alpha} g(t,x)\right| \leqslant \rho_m \int_0^t e^{K'_m s}\,ds, \quad t \geqslant 0,\ x \in \mathbb{R}^n.$$

The proof of Step 1 is concluded if we choose $K_m = \max\{K'_m, \rho_m\}$.

Consider $\mathbb{R}^n = E \oplus E^{\perp}$.

Step 2. For each $m = 0, 1, 2, \ldots$ and $\alpha \in \mathbb{N}^n$ there exist $H_{m,\alpha}, b_{\alpha} > 0$ such that

$$\left|\partial^{\alpha} f\big(\gamma(t,x)\big)\right| \leqslant H_{m,\alpha} e^{(b_{\alpha} - m\mu)t}, \quad t \geqslant 0,\ x \in E. \tag{4.5}$$

In fact, from chain's rule it follows that $\partial^{\alpha} f(\gamma(t,x))$ is a finite sum of terms of the form

$$\partial^{\alpha^1} f\big(\gamma(t,x)\big) p\big(\ldots, \partial^{\beta}\gamma, \ldots\big), \tag{4.6}$$

where $\alpha^1 \in \mathbb{N}^n$, $0 < |\alpha^1| \leqslant |\alpha|$ and p is a monomial of degree $\leqslant |\alpha|$ on the variables $(\partial^{\beta}\gamma)_{0<\beta\leqslant\alpha}$. Hence, to prove the statement in Step 2 it is enough to prove an estimate like (4.5), but replacing $\partial^{\alpha} f(\gamma(t,x))$ by a term of the form (4.6).

Since f has compact support and f is flat in N, it is easy to see that there exists $C_{m,\alpha^1} > 0$ such that

$$\left|\partial^{\alpha^1} f(x)\right| \leqslant C_{m,\alpha^1} |x - N|^m, \quad x \in \mathbb{R}^n.$$

From the hypothesis (1.1) of Theorem 1.4 we obtain

$$\left|\partial^{\alpha^1} f\big(\gamma(t,x)\big)\right| \leqslant C_{m,\alpha^1} M^m e^{-m\mu t} |x - N|^m, \quad t \geqslant 0,\ x \in E. \tag{4.7}$$

On the other hand, in the same way as in the proof of (3.4) and using Lemma 3.3 we prove that there exist $h_{m,\alpha}, b'_{\alpha} > 0$ such that

$$\left|p\big(\ldots, \partial^{\beta}\gamma, \ldots\big)\right| \leqslant h_{m,\alpha} e^{b'_{\alpha} t}, \quad t \geqslant 0,\ x \in \mathbb{R}^n. \tag{4.8}$$

Combining (4.7) and (4.8) we conclude the proof of Step 2.

Step 3. For all $\alpha \in \mathbb{N}^n$ and $x \in E$, $\lim_{t\to+\infty} \partial^{\alpha} v(t,x)$ exists and

$$v^{\alpha}(x) := \lim_{t\to+\infty} \partial^{\alpha} v(t,x), \quad x \in E,$$

belongs to $C^{\infty}(E)$.

In fact, first we will show that $v^0 \in C^1(E)$. Choose $e_j \in E$ and notice that the derivative in x_j of the term under the integral in (4.2) is

$$e^{g(t,x)}\big[\partial_j g(t,x) f\big(\gamma(t,x)\big) + \partial_j\big(f\big(\gamma(t,x)\big)\big)\big].$$

Choose $m=1$ in Step 1. Then use Step 2 with $\alpha=0$, with $\alpha=e_j$ and m satisfying $\rho_0+K_1+b_0-m\mu<0$ and $\rho_0+b_{e_j}-m\mu<0$. From the Dominated Convergence Theorem and (4.3) it follows that $\lim_{t\to+\infty}\partial_j v(t,x)$ exists for all $x\in E$ and belongs to $C^0(E)$. This concludes the proof that $v^0\in C^1(E)$.

Suppose that $v^0\in C^k(E)$. To prove that $v^0\in C^{k+1}(E)$, notice that, for each $\alpha\in\mathbb{N}$ with $\alpha\neq 0$, the α derivative of the term under the integral (4.2) is a finite sum of terms of the form

$$e^{g(t,x)}\partial^{\alpha^1}\big(f\big(\gamma(t,x)\big)\big)q\big(\ldots,\partial^\beta\gamma,\ldots\big),\quad t\geqslant 0,\ x\in\mathbb{R}^n,$$

where $\alpha^1\in\mathbb{N}^n$, $0<|\alpha^1|\leqslant|\alpha|$ and q is a monomial of degree $\leqslant|\alpha|$ on the variables $(\partial^\beta\gamma)_{0<\beta\leqslant\alpha}$. From the induction hypothesis, the estimates of Steps 1 and 2 and the Dominated Convergence Theorem we obtain that $v^0\in C^\infty(E)$.

In the same way we can prove that $v^\alpha\in C^\infty(E)$, $\forall\alpha\in\mathbb{N}^n$. Moreover, for all $\alpha\in\mathbb{N}^n$ and for all β multi-index on the variables of E we have

$$\partial^\beta v^\alpha(x)=v^{\alpha+(\beta,0)}(x),\quad x\in E.$$

Hence Step 3 is finished.

Consider $u\in C^\infty(\mathbb{R}^n)$ such that

$$\partial^\alpha u(y_1,0)=v^\alpha(y_1),\quad y_1\in E,\ \alpha\in\mathbb{N}^n. \tag{4.9}$$

To construct such function we use the recipe given in the proof of Guillemin–Schaeffer's Theorem [3] or as in [8]. Consider

$$u(y_1,y_2)=\sum_\alpha\frac{y_2^\alpha}{\alpha!}v^\alpha(y_1)\theta\big(\big|v^\alpha(y_1)\big|^2|y_2|^2\big).$$

Here $y_1\in E$, $y_2\in E^\perp$, α is a multi-index in the y_2 variable and $\theta:\mathbb{R}\to[0,1]$ is a smooth function which is equal to one near the origin and the support of θ is contained in $[-1,1]$.

Step 4. $Pu-f$ vanishes of infinite order on E.

In fact, from (4.1) it follows that

$$\frac{d}{dt}g(t,x)=c\big(\gamma(t,x)\big),\quad t\geqslant 0,\ x\in\mathbb{R}^n \tag{4.10}$$

and

$$g(0,x)=0,\quad x\in\mathbb{R}^n. \tag{4.11}$$

Moreover, we have that

$$g\big(t,\gamma(s,x)\big)=g(t+s,x)-g(s,x),\quad t,s\geqslant 0,\ x\in\mathbb{R}^n. \tag{4.12}$$

Given $x\in E$ and $s\geqslant 0$, from (4.12) we obtain

$$u\big(\gamma(s,x)\big)=-e^{-g(s,x)}\int_0^{+\infty}e^{g(t+s,x)}f\big(\gamma(t+s,x)\big)\,dt$$

hence

$$u\big(\gamma(s,x)\big) = -e^{-g(s,x)} \int_s^{+\infty} e^{g(t,x)} f\big(\gamma(t,x)\big)\,dt.$$

Differentiating the above identity with respect to s, from (4.10) we have

$$L\big(u\big(\gamma(s,x)\big)\big) = c\big(\gamma(s,x)\big)e^{-g(s,x)} \int_s^{+\infty} e^{g(t,x)} f\big(\gamma(t,x)\big)\,dt + f\big(\gamma(s,x)\big).$$

Take $s=0$ in the above identity, from (4.11) we obtain $Lu = -cu + f$. Using that E is invariant under the flow of L we may repeat this argument for the partial derivatives of u to prove that $Pu - f$ vanishes of infinite order on E. $\square$

Proof of Theorem 1.3. Since the conclusion of Theorem 1.3 is local, from Proposition 2.1 we may suppose that $L \in \mathcal{V}$ and $c, f \in C_0^\infty(\mathbb{R}^n)$. From Remark 2.3 replacing L by $-L$, we may apply Theorem 1.4 with $N = \{0\}$ and $E = W^u(0)$.

Let $u_1 \in C^\infty(\mathbb{R}^n)$ be such that $f_1 = f - Pu_1$ vanishes of infinite order at $W^u(0)$. A second application of Theorem 1.4 with $N = W^u(0)$, $E = \mathbb{R}^n$ and f replaced by f_1 concludes the proof of Theorem 1.3. $\square$

Proof of Theorem 1.2. From now on, let P be as in Theorem 1.2. In [9] we proved the following result.

Theorem 4.1. *Consider X, L, c and x_0 as in Theorem* 1.2. *Suppose that* (b) *and* (c) *of Theorem* 1.2 *hold. Then $\forall f \in C^\infty(X)$ such that $f = 0$ in a neighborhood of x_0 there exists $u \in C^\infty(X)$, with $u = 0$ in a neighborhood of x_0, such that $Pu = f$ in X.*

Combining Lemma 1.1 and Theorem 1.3 we solve the equation $Pu = f$ in a neighborhood of x_0 for an arbitrary $f \in C^\infty(X)$. Using Theorem 4.1 we conclude the proof of Theorem 1.2. $\square$

Appendix A

Before the proof of Lemma 1.1 we prove some preliminary results. Given $A \in M(\mathbb{R}, n)$, $b \in \mathbb{R}$ define $T(A, 0, b) = A$, and for $m \in \mathbb{N}$ define

$$T(A,m,b) = \begin{bmatrix} A & mbI & 0 & \cdots & 0 & 0 \\ -bI & A & (m-1)bI & \cdots & 0 & 0 \\ 0 & -2bI & A & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & A & bI \\ 0 & 0 & 0 & \cdots & -mbI & A \end{bmatrix},$$

where $I \in \mathbb{R}^{n\times n}$ is the identity matrix.

Consider $T_0(m:b) = T(0,m,b)$, where $0 \in \mathbb{R}$ and $m = 0,1,2,\ldots$. Given $m_1, m_2 = 0,1,2,\ldots$ and $b_1, b_2 \in \mathbb{R}$ define $T_0(m_1, m_2 : b_1, b_2) = T(T_0(m_1 : b_1), m_2, b_2)$. Define $T_0(m_1, m_2, \ldots, m_r : b_1, b_2, \ldots, b_r)$ in the same way.

Here $\operatorname{Spec} A$ denotes the set of the eigenvalues of the matrix A.

Lemma A.1. $\operatorname{Spec} T_0(m:b) = \pm ib\mathscr{M}(m)$, $m = 0,1,2,\ldots$.

Proof. Clearly we may suppose that $b=1$. Since $T_0(m-1:1)\in M(\mathbb{R},m)$ we will prove that

$$\operatorname{Spec} T_0(m-1:1)=\pm i\mathscr{M}(m-1),\quad m=1,2,\ldots. \tag{A.1}$$

We denote the characteristic polynomial of $T_0(m-1:1)$ by p_m and

$$p_m(\lambda)=\sum_{k=0}^{m}[p_m]_k\lambda^k,\quad \lambda\in\mathbb{R},\ m=1,2,\ldots,$$

where $[p_m]_k$ is the coefficient of λ^k in p_m.

First consider the case when m is even. Hence, to show (A.1) we will prove that

$$p_{m+2}(\lambda)=p_m(\lambda)\big(\lambda^2+(m+1)^2\big),\quad \lambda\in\mathbb{R},\ m=2,4,\ldots. \tag{A.2}$$

The strategy is to compare the correspondent coefficients of the polynomials in (A.2). Since m is even and $T_0(m-1:1)$ is a tridiagonal matrix with diagonal equal to zero, we observe that $[p_m]_k=0$ when k is odd. Then (A.2) is equivalent to

$$\begin{cases}[p_{m+2}]_{m+2}=[p_m]_m,\\ [p_{m+2}]_k=[p_m]_{k-2}+(m+1)^2[p_m]_k,\quad k=2,4,\ldots,m,\\ [p_{m+2}]_0=(m+1)^2[p_m]_0.\end{cases} \tag{A.3}$$

For $l=1,2,\ldots,\frac{m}{2}$, we can prove that

$$[p_m]_{m-2l}=\sum_{k_1=1}^{m-2l+1}k_1(m-k_1)\sum_{k_2=k_1+2}^{m-2l+3}k_2(m-k_2)\cdots\sum_{k_l=k_{l-1}+2}^{m-1}k_l(m-k_l). \tag{A.4}$$

Define $S_0(m,j)=1$, $j=1,2,\ldots$. Also, for each $l=1,2,\ldots,\frac{m}{2}$ and $j=1,2,\ldots,m-2l+1$ define

$$S_l(m,j)=\sum_{k=j}^{m-2l+1}k(m-k)S_{l-1}(m,k+2).$$

From (A.4) we have $S_l(m,1)=[p_m]_{m-2l}$, $l=1,2,\ldots,\frac{m}{2}$. Since the first identity of (A.3) is trivial, we have that (A.3) is equivalent to

$$\begin{cases}S_l(m+2,1)=S_l(m,1)+(m+1)^2S_{l-1}(m,1),\quad l=1,2,\ldots,\dfrac{m}{2},\\[2mm] S_l(m+2,1)=(m+1)^2S_{l-1}(m,1),\quad l=\dfrac{m}{2}+1.\end{cases} \tag{A.5}$$

To prove (A.5) we use the following properties:

(P1) $S_l(m,m-2l+1)=\prod_{k=1,\,k\text{ odd}}^{2l-1}k(m-k)$, $l=1,2,\ldots,\frac{m}{2}$,
(P2) $S_{l-1}(m,m-2l+5)=(m+1)(2l-3)S_{l-2}(m,m-2l+5)$, $l=1,2,\ldots,\frac{m}{2}$,
(P3) $S_l(m,j)=S_l(m,j+1)+j(m-j)S_{l-1}(m,j+2)$, $l=1,2,\ldots,\frac{m}{2}$, $j=0,1,\ldots,m-2l$,
(P4) $\sum_{k=j}^{m-2l+1}S_l(m,j)=\sum_{k=j}^{m-2l+1}k(m-k)(k-j+1)S_{l-1}(m,k+2)$, $l=1,2,\ldots,\frac{m}{2}$, $j=1,2,\ldots,m-2l+1$,
(P5) $S_l(m+2,j+1)=S_l(m,j)+(m+1)(m-j+1)S_{l-1}(m,j+1)$, $l=1,2,\ldots,\frac{m}{2}$, $j=1,2,\ldots,2l+1$.

The properties (P1)–(P4) are elementary and (P5) follows by induction on l, (P2), and (P4).

Using (P1), (P3) and (P5) we prove (A.5). In fact, we will prove the first identity of (A.5). If $l=1$ then using (P3) with $m+2$ replaced by m and $j=1$ we have $S_1(m+2,1)=S_1(m+2,2)+m+1$. From (P5) it follows that

$$S_1(m+2,1)=S_1(m,1)+(m+1)m+m+1.$$

If $l=2,3,\dots,\frac{m}{2}$, using (P3) with $m+2$ replaced by m and $j=1$ we have

$$S_l(m+2,1)=S_l(m+2,2)+(m+1)S_{l-1}(m+2,3).$$

Using (P5) in the both terms on the right

$$S_l(m+2,1)=S_l(m,1)+(m+1)^2\big[S_{l-1}(m,2)+(m-1)S_{l-2}(m,3)\big].$$

Since $l\geqslant 2$, from (P3) it follows the first identities of (A.5). The last identity follows from (P1).

The proof for the case when m is odd follows in the same way. In fact, we observe that the identity (A.3) is true for $m=1$. To prove (A.3) for $m=3,5,\dots$ we replace $k=2,4,\dots,m$ in (A.3) by $k=3,4,\dots,m$. The rest of the proof follows if we replace $l=1,2,\dots,\frac{m}{2}$ by $l=1,2,\dots,\frac{m-1}{2}$ and $l=\frac{m}{2}$ by $l=\frac{m-1}{2}+1$, respectively. $\square$

Lemma A.2. $\operatorname{Spec} T_0(m_1,m_2,\dots,m_r:b_1,b_2,\dots,b_r)=\pm i\sum_{k=1}^{r} b_k\mathscr{M}(m_k)$.

Proof. We will use induction in r. The case $r=1$ follows from Lemma A.1. Suppose that the result holds for r.

Given matrices $A_1,A_2,\dots,A_k$, let $\operatorname{diag}(A_1,A_2,\dots,A_k)$ be the block diagonal matrix which the entries in the diagonal are $A_1,A_2,\dots,A_k$, respectively.

Let A be a matrix such that $AT_0(m_1,m_2,\dots,m_r:b_1,b_2,\dots,b_r)A^{-1}=D$, where D is a diagonal matrix. Consider $\operatorname{diag}(A,A,\dots,A)$ with $m_{r+1}+1$ blocks in the diagonal. We observe that

$$\begin{aligned}&\operatorname{diag}(A,A,\dots,A)T_0(m_1,m_2,\dots,m_{r+1}:b_1,b_2,\dots,b_{r+1})\operatorname{diag}\big(A^{-1},A^{-1},\dots,A^{-1}\big)\\&\quad=T(D,m_{r+1},b_{r+1}).\end{aligned}$$

We can prove that the characteristic polynomial of the matrix on the right is

$$p(\lambda-\lambda_1)p(\lambda-\lambda_2)\cdots p(\lambda-\lambda_l),$$

where p is the characteristic polynomial of $T_0(m_{r+1}:b_{r+1})$ and $\lambda_1,\lambda_2,\dots,\lambda_l$ are the eigenvalues of D. The proof follows from the induction hypothesis and Lemma A.1. $\square$

Proof of Lemma 1.1. We may suppose that $X=\mathbb{R}^n$ and $x_0=0$. We denote by $Pu\sim f$ when $Pu-f$ is flat at the origin. Write $L=\sum_{j=1}^{n}a_j\partial_j$ and consider the formal Taylor expansions of u, a_j and c at $x=0$:

$$\begin{aligned}&\sum_\alpha\frac{\partial^\alpha u(0)}{\alpha!}x^\alpha,\\&\sum_\alpha\frac{\partial^\alpha a_j(0)}{\alpha!}x^\alpha,\quad j=1,2,\dots,n,\\&\sum_\alpha\frac{\partial^\alpha c(0)}{\alpha!}x^\alpha,\end{aligned}$$

respectively. Then $Pu \sim f$ is equivalent to

$$\sum_{j,k} \alpha_k \partial_k a_j(0) \partial^{\alpha+e_j-e_k} u(0) + c(0)\partial^\alpha u(0) + R_\alpha = \partial^\alpha f(0), \quad \forall \alpha \in \mathbb{N}^n, \tag{A.6}$$

where e_j is the unit vector of $\mathbb{R}^n$ with 1 in the jth position. The term R_α depends only on the derivatives of u of order $\leqslant 1$ evaluated at the origin and has the following property: if $\partial^\beta u(0) = 0$, $\forall \beta \in \mathbb{N}^n$ such that $|\beta| \leqslant |\alpha| - 1$, then $R_\alpha = 0$.

Hence, $Pu \sim f$ is equivalent to a sequence of linear systems

$$\bigl(B^m + c(0)I\bigr)u^m = f^m + v^{m-1}, \quad m \in \mathbb{N}. \tag{A.7}$$

Consider $\Lambda_n^m = \{\alpha \in \mathbb{N}^n;\ |\alpha| = m\}$ and $M = \sharp \Lambda_n^m$. For each $m \in \mathbb{N}$, B^m is a real matrix $M \times M$ which depends on $DL(0)$ and on the choice of an ordering of Λ_n^m. The components of $u^m \in \mathbb{C}^M$ (resp. $f^m \in \mathbb{C}^M$) are the derivatives of u (resp. f) of order m evaluated at the origin. If $m \geqslant 1$ then the vector $v^{m-1} \in \mathbb{C}^M$ corresponds to the term R_α of (A.6). Define $v^0 = 0 \in \mathbb{R}$. The vector v^{m-1} depends only on the derivatives of u of order $\leqslant m-1$ and this vector has the following property:

$$\partial^\alpha u(0) = 0, \quad \forall \alpha \in \mathbb{N}^n \text{ satisfying } |\alpha| \leqslant m-1 \quad \Rightarrow \quad v^{m-1} = 0. \tag{A.8}$$

From (A.8) it follows that the system (A.7) can be solved recursively for $u^0, u^1, \ldots,$ if, and only if, $-c(0) \notin \operatorname{Spec}(B^m)$. To conclude the proof we will compute $\operatorname{Spec}(B^m)$.

Define $r = (n - n')/2$. Using the real Jordan form of $DL(0)$ and a suitable choice of ordering for Λ_n^m we obtain that B^m is an upper triangular matrix. Moreover, each block of the diagonal is of the form

$$\left(\sum_{j=1}^{n'} m'_j \lambda_j + \sum_{j=1}^{r} m_j \operatorname{Re} \lambda_{n'+2j-1}\right) I + T_0(m_1, m_2, \ldots, m_r : \operatorname{Im} \lambda_{n'+1}, \operatorname{Im} \lambda_{n'+3}, \ldots, \operatorname{Im} \lambda_{n-1}),$$

with $m'_j, m_j \in \mathbb{N}$ and

$$m'_1 + m'_2 + \cdots + m'_{n'} + m_1 + m_2 + \cdots + m_r = m.$$

Hence, the proof of Lemma 1.1 follows from Lemma A.2. $\square$

Appendix B

In this appendix we present an alternative proof of Lemma A.1 using the representation theory of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. For simplicity, let us assume $b = 1$ and represent the matrix $T_0(m : b)$ simply by

$$A_m = \begin{bmatrix} 0 & m & 0 & \cdots & 0 & 0 \\ -1 & 0 & m-1 & & 0 & 0 \\ 0 & -2 & 0 & & 0 & 0 \\ 0 & 0 & -3 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & & 0 & 1 \\ 0 & 0 & 0 & \cdots & -m & 0 \end{bmatrix}.$$

Obviously, multiplying A_m by $b \in \mathbb{R}$ has the effect of multiplying its eigenvalues by b, so there is no loss of generality in our assumption. Furthermore, in order to use only simple facts from representation theory, we find it slightly more convenient to work with the matrices

$$B_m = \begin{bmatrix} 0 & m & 0 & \cdots & 0 & 0 \\ 1 & 0 & m-1 & & 0 & 0 \\ 0 & 2 & 0 & & 0 & 0 \\ 0 & 0 & 3 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & & 0 & 1 \\ 0 & 0 & 0 & \cdots & m & 0 \end{bmatrix}.$$

In fact, the following interesting relationship between the eigenvalues of these matrices holds:

Lemma B.1. *Let A_m and B_m be as above. Then $\lambda \in \mathbb{C}$ is an eigenvalue of B_m if, and only if $i\lambda$ is an eigenvalue of A_m.*

Proof. It is useful to compute the characteristic polynomial $p_{C_m}(\lambda)$ of the slightly more general matrix

$$C_m = \begin{bmatrix} 0 & x_1 & 0 & \cdots & 0 & 0 \\ y_1 & 0 & x_2 & & 0 & 0 \\ 0 & y_2 & 0 & & 0 & 0 \\ 0 & 0 & y_3 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & & 0 & x_m \\ 0 & 0 & 0 & \cdots & y_m & 0 \end{bmatrix},$$

where the x_j and y_j are indeterminates. Using the permutation expansion of the determinant, we have

$$\begin{aligned} p_{C_m}(\lambda) &= \det(\lambda I - C_m) \\ &= \sum_{\sigma \in S_m} \epsilon(\sigma)(\delta_{1\sigma(1)}\lambda - c_{1\sigma(1)}) \cdots (\delta_{m\sigma(m)}\lambda - c_{m\sigma(m)}), \end{aligned} \tag{B.1}$$

where S_m is the symmetric group of degree m, $\epsilon(\sigma) = \pm 1$ is the sign of the permutation σ and $\delta \in \{0, 1\}$ is the Kronecker delta.

Now, due to the arrangement of the zeros in C_m, it is immediate that $c_{j\sigma(j)} = 0$, unless $|\sigma(j) - j| \leqslant 1$, thus the only possibly non-zero terms in (B.1) are those that satisfy $|\sigma(j) - j| \leqslant 1$ for all j. Furthermore, since σ is bijective, it follows at once that such σ factors as a product of disjoint transpositions

$$\sigma = (j_1\ j_1 + 1)(j_2\ j_2 + 1) \cdots (j_r\ j_r + 1)$$

where $j_1, j_1 + 1, j_2, j_2 + 1, \ldots, j_r, j_r + 1$ are all distinct. Therefore, we may rewrite (B.1) as

$$\begin{aligned} p_{C_m}(\lambda) &= \sum_{(j_1,\ldots,j_r)} (-1)^r (-c_{j_1,j_1+1})(-c_{j_1+1,j_1}) \cdots (-c_{j_r,j_r+1})(-c_{j_r+1,j_r})\lambda^{m+1-2r} \\ &= \sum_{(j_1,\ldots,j_r)} (-1)^r x_{j_1} y_{j_1} \cdots x_{j_r} y_{j_r} \lambda^{m+1-2r} \end{aligned} \tag{B.2}$$

where the sum is over all subsets $\{j_1, \ldots, j_r\}$ such that $j_1, j_1+1, \ldots, j_r, j_r+1$ are all distinct together with the empty set. In particular, this shows that all the exponents in p_{C_m} have the same parity, and the same holds for p_{A_m} and p_{B_m}. The statement of the lemma now follows at once from this observation. □

Before we state and prove the main result, we recall that the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ can be realized as the subspace of the 2×2 complex matrices spanned by

$$x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

where the bracket operation is the commutator: $[a, b] = ab - ba$. These satisfy the defining relations

$$[h, x] = 2x, \qquad [x, y] = h, \qquad [h, y] = -2y. \tag{B.3}$$

It is well known [5] that if $\rho : \mathfrak{sl}_2(\mathbb{C}) \to \mathrm{End(V)}$ is a finite-dimensional complex irreducible representation, then the eigenvalues of $\rho(h)$ are the integers $m, m-2, \ldots, 2-m, -m$ where $\dim(V) = m+1$ and the eigenvectors of $\rho(h)$ form a basis $\{v_0, v_1, \ldots, v_m\}$ of V such that $\rho(x)(v_0) = 0$, and $v_j = (1/j!)\rho(y)^j(v_0)$ for all $j > 0$. With respect to this basis we have (Humphreys 1972)

$$\begin{aligned} \rho(x)(v_j) &= (m - j + 1)v_{j-1}, \\ \rho(h)(v_j) &= (m - 2j)v_j, \\ \rho(y)(v_j) &= (j + 1)v_{j+1}. \end{aligned} \tag{B.4}$$

The main result in this appendix is the following:

Theorem B.1. *For each $m \geqslant 1$ the eigenvalues of A_m are precisely the $m+1$ imaginary numbers im, $i(m-2)$, $\ldots$, $i(2-m)$, $-im$.*

Proof. From the previous lemma, we only need to show that the eigenvalues of B_m are the $m+1$ integers $m, m-2, \ldots, 2-m, -m$. We shall show this by constructing an irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$ in which the element h is represented by B_m.

Let $\rho : \mathfrak{sl}_2(\mathbb{C}) \to \mathrm{End\,(V)}$ be the (unique) irreducible complex representation of dimension $m+1$, and let $\sigma : \mathfrak{sl}_2(\mathbb{C}) \to \mathrm{End(V)}$ be the linear transformation given by

$$\begin{aligned} \sigma(x) &= \frac{1}{2}\big(\rho(y) - \rho(x) + \rho(h)\big), \\ \sigma(h) &= \rho(x) + \rho(y), \\ \sigma(y) &= \frac{1}{2}\big(\rho(x) - \rho(y) + \rho(h)\big). \end{aligned}$$

It is a straightforward calculation, using relations (B.3) and the fact that ρ is a representation, to check that $\sigma([a, b]) = [\sigma(a), \sigma(b)]$ for all $a, b \in \mathfrak{sl}_2(\mathbb{C})$, hence σ is also a representation. Furthermore, due to relations (B.4), in the basis $\{v_0, v_1, \ldots, v_m\}$ of V, $\sigma(h)$ is represented by the matrix B_m. It only remains to show that σ is irreducible.

Consider $w_0 = v_0 + v_1 + \cdots + v_m$. A direct calculation shows that $\sigma(x)(w_0) = 0$ and $\sigma(h)(w_0) = mw_0$. On the other hand, Weyl's Theorem [5] asserts that if σ is completely reducible hence it must contain an irreducible subrepresentation accounting for the eigenvalue m. But the eigenvalues of an irreducible representation form a chain, like $k, k-1, \ldots, 2-k, -k$, therefore such irreducible that accounts for m must have at least $m+1$ which is the dimension of V, so σ is irreducible. This completes the proof. □

The proof also shows that the eigenvectors of B_m are the vectors $\{w_0, w_1, \ldots, w_m\}$, where $w_j = (1/j!)\sigma(y)^j(w_0)$, $j = 1, 2, \ldots, m$.

References

[1] F. Braun, J.R. dos Santos Filho, The real Jacobian conjecture on $\mathbb{R}^2$ is true when one of the components has degree 3, Discrete Contin. Dyn. Syst. 26 (2010) 75–87.
[2] J. Duistermaat, L. Hörmander, Fourier integral operators II, Acta Math. 128 (1972) 183–269.
[3] V. Guillemin, D. Schaeffer, On a certain class of Fuchsian partial differential equations, Duke Math. J. 44 (1977) 157–199.
[4] P. Hartman, Ordinary Differential Equations, John Wiley and Sons, New York, 1964.
[5] J.E. Humphreys, Introduction to Lie Algebras and Representation Theory, Springer-Verlag, New York, 1972.
[6] B. Malgrange, Existence et approximation des solutions des équations aus dérivées partielles et des équations de convolution, Ann. Inst. Fourier (Grenoble) 6 (1955–1956) 271–355.
[7] W. Melo, J. Palis, Geometric Theory of Dynamical Systems, Springer-Verlag, New York, 1982.
[8] E. Nelson, Topics in Dynamics I: Flows, Princeton University Press, Princeton, 1969.
[9] J.R. dos Santos Filho, M.F. Silva, Global solvability for first order real linear partial differential operators, J. Differential Equations 247 (2009) 2688–2704.
[10] J.R. dos Santos Filho, J. Tavares, Injective mappings and solvable vector fields, An. Acad. Brasil. Ciênc. 82 (2010) 1–5.
[11] S. Sternberg, On the structure of local homeomorphisms of Euclidean n-space II, Amer. J. Math. 80 (1958) 623–632.